HERMITE FUNCTIONS AND UNCERTAINTY PRINCIPLES FOR THE FOURIER AND THE WINDOWED FOURIER TRANSFORMS

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ABSTRACT. We extend an uncertainty principle due to Beurling into a characterization of Hermite functions. More precisely, all functions f on $\mathbb{R}^d$ which may be written as $P(x)\exp(Ax, x)$, with A a real symmetric definite positive matrix, are characterized by integrability conditions on the product $f(x)\widehat{f}(y)$. We also give the best constant in uncertainty principles of Gelf'and Shilov type. We then obtain similar results for the windowed Fourier transform (also known, up to elementary changes of functions, as the radar ambiguity function or the Wigner transform). We complete the paper with a sharp version of Heisenberg's inequality for this transform.

1. INTRODUCTION AND NOTATIONS.

Uncertainty principles state that a function and its Fourier transform cannot be simultaneously sharply localized. To be more precise, let $d \geq 1$ be the dimension, and let us denote by $\langle .,. \rangle$ the scalar product and by $\|.\|$ the Euclidean norm on $\mathbb{R}^d$. Then, for $f \in L^2(\mathbb{R}^d)$, define the Fourier transform of f by

$$\widehat{f}(y) = \int_{\mathbb{R}^d} f(t)e^{-2i\pi \langle t,y \rangle} dt.$$

The most famous uncertainty principle, due to Heisenberg and Weil, can be stated in the following directional version:

Heisenberg's inequality. Let $i = 1, \ldots, d$ and $f \in L^2(\mathbb{R}^d)$. Then

$$(1) \qquad \inf_{a \in \mathbb{R}} \left(\int_{\mathbb{R}^d} (x_i - a)^2 |f(x)|^2 dx \right) \inf_{b \in \mathbb{R}} \left(\int_{\mathbb{R}^d} (\xi_i - b)^2 \left| \widehat{f}(\xi) \right|^2 d\xi \right) \geq \frac{\|f\|_{L^2}^4}{16\pi^2}.$$

Moreover (1) is an equality if and only if f is of the form

$$f(x) = C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)e^{-2i\pi b x_i}e^{-\alpha(x_i - a)^2}$$

where C is a function in $L^2(\mathbb{R}^{d-1})$, $\alpha > 0$, and a and b are real constants for which the two infimums in (1) are realized.

The usual non-directional uncertainty principle follows easily from this one. We refer to the recent survey articles by Folland and Sitaram [11] and Dembo, Cover and Thomas [9] as well as the book of Havin and Jöricke [15] for various uncertainty principles of different nature which may be found in the literature. One theorem stated in [11] is due to Beurling. Its proof has been written much later by Hörmander in [16]. Our first aim is to weaken the assumptions so that non zero solutions given by Hermite functions are also possible. More precisely, we will prove the following theorem:

Theorem 1.1 (Beurling-Hörmander type). *Let $f \in L^2(\mathbb{R}^d)$ and $N \geq 0$. Then*

$$(2) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi|\langle x, y \rangle|} dx dy < +\infty$$

if and only if f may be written as

$$f(x) = P(x)e^{-\pi \langle Ax, x \rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial of degree $< \frac{N-d}{2}$.

In particular, for $N \leq d$, the function f is identically 0. Beurling-Hörmander's original theorem is the above theorem for $d = 1$ and $N = 0$. An extension to $d \geq 1$ but still $N = 0$ has been given, first by S.C. Bagchi and S. K. Ray in [2] in a weaker form, then very recently by S. K. Ray and E. Naranayan in the present form. Their proof, which relies on the one dimensional case, uses Radon transform [23].

Let us remark that it is easy to adapt the usual proof of Hardy's Theorem to allow Hermite functions (see for instance [22]), while the proof given here for integrability conditions uses new ingredients. At the same time, it simplifies Hörmander's argument for the case $N = 0$, $d = 1$, in such a way that the proof can now be given in any textbook on Fourier Analysis. We give the simplified proof in the Appendix, since it may be useful in this context.

The previous theorem has as an immediate corollary the following characterization.

Corollary 1.2. *A function $f \in L^2(\mathbb{R}^d)$ may be written as*

$$f(x) = P(x)e^{-\pi \langle Ax, x \rangle},$$

with A a real positive definite symmetric matrix and P a polynomial, if and only if the function

$$f(x)\widehat{f}(y) \exp(2\pi|\langle x, y \rangle|)$$

is slowly increasing on $\mathbb{R}^d \times \mathbb{R}^d$.

As an easy consequence of the previous theorem, we also deduce the following corollary, which generalizes the Cowling-Price uncertainty principle (see [8]).

Theorem 1.3 (Cowling-Price type). *Let $N \geq 0$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)| \frac{e^{\pi a|x_j|^2}}{(1 + |x_j|)^N} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\widehat{f}(y)| \frac{e^{\pi b|y_j|^2}}{(1 + |y_j|)^N} dy < +\infty$$

for $j = 1, \cdots, d$ and for some positive constants a and b with $ab = 1$. Then $f(x) = P(x)e^{-a\|x\|^2}$ for some polynomial P.

The Cowling-Price theorem is given in [8] for $d = 1$, $N = 0$, and with p-th powers as well. This last extension is a trivial consequence of Hölder's inequality once N is allowed to take positive values.

It is remarked in [16], as well as in [2], that a theorem of Beurling-Hörmander type implies also a theorem of Gel'fand-Shilov type. In fact, the constant that one obtains when doing this is not the best one, and looking carefully at the literature on entire functions one gets the following theorem, which gives the critical constant.

Theorem 1.4 (Gel'fand-Shilov type). *Let $1 < p < 2$, and let q be the conjugate exponent. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)| e^{2\pi \frac{a^p}{p}|x_j|^p} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\widehat{f}(y)| e^{2\pi \frac{b^q}{q}|y_j|^q} dy < +\infty$$

for some $j = 1, \cdots, d$ and for some positive constants a and b. Then $f = 0$ if $ab > \left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$. If $ab < \left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$, one may find a dense subset of functions which satisfy the above conditions for all j.

From Theorem 1.1, one only gets the first result for $ab \geq 1$, which is clearly much weaker. Close but different statements are given in [15].

One way one may hope to overcome the lack of localisation is to use the *windowed Fourier transform*, also known as the (continuous) *Gabor transform* or the *short-time Fourier transform*. To be more precise, fix $v \in L^2(\mathbb{R}^d)$, the "window", and define for $u \in L^2(\mathbb{R}^d)$:

$$S_v u(x,y) = \overline{\widehat{\overline{u}v(.-x)}}(y) = \int_{\mathbb{R}^d} u(t)\overline{v(t-x)}e^{2i\pi\langle t,y\rangle}dt.$$

This transform occurs also in several other forms, for example $|S_v u|^2$ is known as a *spectrogram*. For sake of symmetry in u and v, we rather focus on the *radar ambiguity function* defined for $u, v \in L^2(\mathbb{R}^d)$ by

$$(3) \qquad A(u,v)(x,y) = \int_{\mathbb{R}^d} u\left(t+\frac{x}{2}\right)\overline{v\left(t-\frac{x}{2}\right)}e^{-2i\pi\langle y,t\rangle}dt.$$

Since $|A(u,v)| = |S_v u|$, there will be no loss in doing so. We refer the reader to [1], [7], [6] and the references there for the way these functions occur in signal processing, and their basic properties.

Finally, $W(u,v)$, the Fourier transform of $A(u,v)$ in $\mathbb{R}^{2d}$ is known in quantum mechanics and in the PDE community as the *Wigner transform* or *Wigner distribution*. Since

$$W(u,v)(x,y) = \int_{\mathbb{R}^d} u\left(x+\frac{t}{2}\right)\overline{v\left(x-\frac{t}{2}\right)}e^{2i\pi\langle y,t\rangle}dt,$$

$W(u,v)$ is also related to $A(u,v)$ by

$$W(u,v)(x,y) = 2^d A(u,Zv)(2x,-2y)$$

where $Zv(x) = v(-x)$. So again, all results stated here can be restated in terms of the Wigner transform.

Our second aim here is to extend uncertainty principles to the radar ambiguity functions. In particular, we will show that $A(u,v)$ satisfies theorems of Cowling-Price type on one side, of Gel'fand-Shilov type on the other one. Both results are sharp, with the same characterization of the Hermite functions in the first case.

We also give a version of Heisenberg's inequality for $A(u,v)$ that is stronger than a previous version by A.J.E.M. Janssen ([19], see also [10]). The one dimensional case for the Wigner transform $W(u,u)$ can be found in [3]. This Heisenberg's inequality may be stated in the following matricial form.

Theorem 1.5. *Assume that u and v be in $L^2(\mathbb{R}^d)$, with $\|u\|_{L^2}\|v\|_{L^2}1$ and*

$$\int_{R^d} \|x\|^2(|u|^2 + |v|^2 + |\widehat{u}|^2 + |\widehat{v}|^2)dx < \infty.$$

Let (X,Y) be a random vector with probability density given on $\mathbb{R}^d \times \mathbb{R}^d$ by the function $|A(u,v)|^2$. Then X and Y are not correlated, and their two covariance matrices are such that

$$4\pi^2 V(X) - V(Y)^{-1}$$

is semi-positive definite. Moreover, if it is the zero matrix, then u and v are Gaussian functions.

In Radar Theory, for which $d = 1$, the couple (X,Y) may be given a physical meaning: its first component is related to the distance to the target, the second to its velocity. So the variance gives the quadratic error when estimating the distance or the velocity by the corresponding mean.

A different problem consists in minimizing the same quantity $A(u,v)$ for a fixed "window" v, or more generally to consider uncertainty principles in terms of a function u and its windowed Fourier transform $A(u,v)$. Such problems have been considered by E. Wilczock [25]. Such results follow from ours only when the window is Gaussian.

The article is organized as follows: the next section is devoted to the proof of Theorem 1.1, whereas in section 3, we consider the other mentioned uncertainty principles for the Fourier transform. In section 4, we recall some basic properties of the ambiguity functions, pursuing in section 5 with the Heisenberg inequality for these functions. Section 6 is devoted to the extension of uncertainty principles to ambiguity functions.

2. Generalization of Beurling-Hörmander's theorem.

The statement of Theorem 1.1 is divided into two propositions. The first one is elementary.

Proposition 2.1. *Let $f \in L^2(\mathbb{R}^d)$ be a function of the form*

$$f(x) = P(x)e^{-\pi\langle(A+iB)x,x\rangle},$$

with A and B two real symmetric matrices and P a polynomial. Then the matrix A is positive definite. Moreover, the three following conditions are equivalent:

(i) $B = 0$ and $\deg(P) < \frac{N-d}{2}$;

(ii) $\displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi|\langle x,y\rangle|} dx dy < +\infty$;

(iii) $\displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\|)^{\frac{N}{2}}(1 + \|y\|)^{\frac{N}{2}}} e^{2\pi|\langle x,y\rangle|} dx dy < +\infty$.

Proof. The fact that A is positive definite is elementary. Then, after a change of variables, we may assume that A is the identity matrix, so that f may be written as $P(x)e^{-\pi\|x\|^2}e^{-i\pi\langle Bx,x\rangle}$. The Fourier transform of f may be written as $Q(y)e^{-\pi\langle(I+iB)^{-1}y,y\rangle}$, with $\deg(P) = \deg(Q) = n$.

To prove that (i) implies (iii), it is sufficient to prove that, for $\alpha > \frac{d}{2}$, we have the inequality

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (1 + \|x\|)^{-\alpha}(1 + \|y\|)^{-\alpha} e^{-\pi\|x-y\|^2} dx dy < +\infty .$$

But this integral is twice the integral on the subset where $\|x\| \leq \|y\|$. So it is bounded by

$$2\int_{\mathbb{R}^d} \left[(1 + \|x\|)^{-2\alpha} \times \int_{\mathbb{R}^d} e^{-\pi\|x-y\|^2} dy\right] dx < +\infty$$

which allows to conclude.

It is clear that (iii) implies (ii) for all functions f. Let us now prove that (ii) implies (i). First, writing $(1+iB)^{-1} = (I-iB)(I+B^2)^{-1}$, it is immediate that the integrability of $f(x)\widehat{f}(y)\frac{\exp 2\pi|\langle x,y\rangle|}{(1+\|x\|+\|y\|)^N}$ implies that the homogeneous polynomial $\|x\|^2 + \langle(I + B^2)^{-1}y, y\rangle - 2\langle x,y\rangle$ is always positive, which implies that $B = 0$. Now, let P_n and Q_n be the homogeneous terms of maximal degree of the polynomials P and Q. There exists $x^{(0)} \in \mathbb{R}^d$ of norm 1 such that $P_n(x^{(0)})Q_n(x^{(0)})$ is different from 0. We call Γ_ε the cone which is obtained as the image of the cone

$$\left\{x = (x_1, \cdots, x_d) ; (x_2^2 + \cdots + x_d^2)^{\frac{1}{2}} < \varepsilon x_1\right\}$$

under a rotation which maps the point $(1, 0, \cdots, 0)$ to $x^{(0)}$. Then, for $\varepsilon < 1$ small enough, there exists a constant c such that, for $x, y \in \Gamma_\varepsilon$, one has the inequality

$$|P_n(x)||Q_n(y)| \geq c\|x\|^n\|y\|^n .$$

The same inequality is valid for P and Q for x and y large, which implies that

$$\iint_{\Gamma_\varepsilon \times \Gamma_\varepsilon} \frac{\|x\|^n\|y\|^n}{(1 + \|x\| + \|y\|)^N} e^{-\pi\|x-y\|^2} dx dy < +\infty .$$

We remark that if $x \in \Gamma_\varepsilon$ then $-\Gamma_\varepsilon \subset x - \Gamma_\varepsilon$. So, *a fortiori*, we have that

$$\iint_{\Gamma_\varepsilon \times (-\Gamma_\varepsilon)} \frac{\|x\|^n \|y\|^n}{(1 + \|x\| + \|t + x\|)^N} e^{-\pi \|t\|^2} dx dt < +\infty \,.$$

We know, using Fubini's theorem, that there exists t for which

$$\int_{\Gamma_\varepsilon} \frac{\|x\|^n \|y\|^n}{(1 + \|x\| + \|t + x\|)^N} dx < +\infty \,,$$

which proves that $N - 2n > d$.   □

We have written here the equivalence between Conditions (ii) and (iii) to remark that Condition (iii) could be written in place of (2) in Theorem 1.1.

The next proposition is much deeper.

Proposition 2.2. *Let $f \in L^2(\mathbb{R}^d)$. If, for some positive integer N,*

$$(4) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi |\langle x, y \rangle|} dx dy < +\infty \,,$$

then f may be written as

$$f(x) = P(x) e^{-\pi \langle (A + iB)x, x \rangle},$$

where A and B are two symmetric matrices and P is a polynomial.

Proof. We may assume that $f \neq 0$.

First step. Both f and $\widehat{f}$ are in $L^1(\mathbb{R}^d)$.

For almost every y,

$$|\widehat{f}(y)| \int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + \|x\|)^N} e^{2\pi |\langle x, y \rangle|} dx < +\infty.$$

As $f \neq 0$, the set of all y's such that $\widehat{f}(y) \neq 0$ has positive measure. In particular, there is a basis $y^1 \ldots y^d$ of $\mathbb{R}^d$ such that, for $i = 1, \ldots, d$, $\widehat{f}(y^{(i)}) \neq 0$ and

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + \|x\|)^N} e^{2\pi |\langle x, y^{(i)} \rangle|} dx < +\infty.$$

Since, clearly, there exists a constant C such that

$$(1 + \|x\|)^N \leq C \sum_{i=1}^{d} \exp \left[2\pi \left| \left\langle x, y^{(i)} \right\rangle \right| \right],$$

we get $f \in L^1(\mathbb{R}^d)$. Exchanging the roles of f and $\widehat{f}$, we get $\widehat{f} \in L^1(\mathbb{R}^d)$.   ◇

Second step. The function g defined by $\widehat{g}(y) = \widehat{f}(y) e^{-\pi \|y\|^2}$ satisfies the following properties (with C depending only on f)

$$(5) \qquad \int_{\mathbb{R}^d} |\widehat{g}(y)| e^{\pi \|y\|^2} dy < \infty \,;$$

$$(6) \qquad |\widehat{g}(y)| \leq C e^{-\pi \|y\|^2} \,;$$

$$(7) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|g(x)||\widehat{g}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi |\langle x, y \rangle|} dx dy < +\infty \,;$$

$$(8) \qquad \int_{\|x\| \leq R} \int_{\mathbb{R}^d} |g(x)||\widehat{g}(y)| e^{2\pi |\langle x, y \rangle|} dx dy < C(1 + R)^N \,.$$

Property (5) is obvious from the definition of g and the fact that $\widehat{f}$ is in $L^1(\mathbb{R}^d)$. As $f \in L^1(\mathbb{R}^d)$, $\widehat{f}$ is bounded thus (6) is also obvious. To prove (7), we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|g(x)||\widehat{g}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi|\langle x,y \rangle|} dxdy \leq \iint |f(t)||\widehat{f}(y)| A(t,y) e^{2\pi|\langle t,y \rangle|} dtdy$$

with

$$A(t,y) = \int \frac{e^{-\pi\|x\|^2 - \pi\|y\|^2 + 2\pi|\langle x,y \rangle|}}{(1 + \|t - x\| + \|y\|)^N} dx.$$

We claim that

(9)
$$A(t,y) \leq C(1 + \|t\| + \|y\|)^{-N},$$

which allows to conclude. Indeed, separating the cases of $\langle x, y \rangle$ being positive or negative, we get

$$A(t,y) \leq \int \frac{e^{-\pi\|x-y\|^2}}{(1 + \|t - x\| + \|y\|)^N} dx + \int \frac{e^{-\pi\|x+y\|^2}}{(1 + \|t + x\| + \|y\|)^N} dx$$
$$= I_1 + I_2.$$

As $I_2(x,t) = I_1(-x,t)$, it is enough to get a bound for I_1. Now, fix $0 < c < 1$ and write $B = (1 + \|t\| + \|y\|)$, then

$$I_1 \leq \int_{\|x-y\|>cB} e^{-\pi\|x-y\|^2} dx + \int_{\|x-y\|\leq cB} \frac{e^{-\pi\|x-y\|^2}}{(1 + \|t - x\| + \|y\|)^N} dx.$$

We conclude directly for the first integral. For the second one, it is sufficient to note that, if $\|x - y\| \leq c(1 + \|t\| + \|y\|)$, then

$$1 + \|t - x\| + \|y\| \geq 1 + \frac{1}{2}\|t\| + \frac{1}{2}\|y\| - \frac{1}{2}\|x - y\|$$
$$\geq \frac{(1-c)}{2}(1 + \|t\| + \|y\|).$$

This completes the proof of (9) and (7).

Let us finally prove (8). Fix $c > 2$. Then the left hand side is bounded by

$$\int_{\|x\|\leq R} |g(x)| \left(\int_{\|y\|>cR} |\widehat{g}(y)| e^{2\pi|\langle x,y \rangle|} dy + \int_{\|y\|<cR} |\widehat{g}(y)| e^{2\pi|\langle x,y \rangle|} dy \right) dx$$
$$\leq \int_{\|x\|\leq R} |g(x)| \left(\int_{\|y\|>cR} C e^{-(\pi - 2\frac{\pi}{c})\|y\|^2} dy + \int_{\|y\|<cR} |\widehat{g}(y)| e^{2\pi|\langle x,y \rangle|} dy \right) dx$$
$$\leq K\|g\|_{L^1} + \int_{\|x\|\leq R} \int_{\|y\|<cR} |g(x)||\widehat{g}(y)| e^{2\pi|\langle x,y \rangle|} dxdy.$$

Then, if we multiply and divide by $(1 + \|x\| + \|y\|)^N$ in the integral of right side, we get the required inequality (8). This completes the proof of the claim. $\diamond$

Third step. The function g admits an holomorphic extension to $\mathbb{C}^d$ that is of order 2. Moreover, there exists a polynomial R such that for all $z \in \mathbb{C}^d$, $g(z)g(iz) = R(z)$.

It follows from (6) and Fourier inversion that g admits an holomorphic extension to $\mathbb{C}^d$ which we again denote by g. Moreover,

$$|g(z)| \leq C e^{\pi\|z\|^2},$$

with C the L^1 norm of $\widehat{g}$. It follows that g is of order 2. On the other hand, for all $x \in \mathbb{R}^d$ and $e^{i\theta}$ of modulus 1,

$$(10) \qquad |g(e^{i\theta}x)| \leq \int_{\mathbb{R}^d} |\widehat{g}|(y)e^{2\pi|\langle x,y\rangle|}dy.$$

Let us now define a new function G on $\mathbb{C}^d$ by:

$$G : z \to \int_0^{z_1} \cdots \int_0^{z_d} g(u)g(iu)du.$$

As g is entire of order 2, so is G. By differentiation of G, the proof of this step is complete once we show that G is a polynomial.

To do so, we will use (8) and an elementary variant of Phragmèn-Lindelhöf's principle which we recall here, and which may be found in [14]: *let ϕ be an entire function of order 2 in the complex plane and let $\alpha \in\ ]0, \pi/2[$; assume that $|\phi(z)|$ is bounded by $C(1 + |z|)^N$ on the boundary of some angular sector $\{re^{i\beta} : r \geq 0, \beta_0 \leq \beta \leq \beta_0 + \alpha\}$. Then the same bound is valid inside the angular sector (when replacing C by $2^N C$).*

Let us fix a vector $\xi \in \mathbb{R}^d$ and define the function G_ξ on $\mathbb{C}$ by $G_\xi(z) = G(z\xi)$. Then G_ξ is an entire function of order 2 which has polynomial growth on $\mathbb{R}$ and on $i\mathbb{R}$ by (10) and (8). We cannot directly apply Phragmèn-Lindelhöf's principle since we are not allowed to do so on angular sectors of angle $\pi/2$. But, to prove that G has polynomial growth in the first quadrant, it is sufficient to prove uniform estimates of this type inside all angular sectors $\{re^{i\beta} : r \geq 0, 0 < \beta_0 \leq \beta \leq \pi/2\}$. Moreover, it is sufficient to have uniform estimates for the functions $G_\xi^{(\alpha)}(z) = G^{(\alpha)}(z\xi)$, with $0 < \alpha < \beta_0$, and

$$G^{(\alpha)}(z) = \int_0^{z_1} \cdots \int_0^{z_d} g(e^{-i\alpha}u)g(iu)du.$$

$G_\xi^{(\alpha)}$ clearly has polynomial growth on $e^{i\alpha}\mathbb{R}$ and on $i\mathbb{R}$, that is

$$G_\xi^{(\alpha)}(z) \leq C(1 + |x|\|\xi\|)^N.$$

The constant C, which comes from the constant in (8), is independent of α. The same estimate is valid inside the angular sector by the Phragmèn-Lindelhöf's principle, and extends to G_ξ, which we wanted to prove.

Proceeding in an analogous way in the three other quadrants, we prove that G_ξ is an entire function with polynomial growth of order N, so a polynomial of degree $\leq N$. Let us now write

$$G_\xi(z) = a_0(\xi) + \cdots + a_N(\xi)z^N.$$

Then

$$a_j(\xi) = \frac{1}{j!}\frac{d^j}{dz^j}\big(G(z\xi)\big)\big|_{z=0}$$

shows that a_j is a homogeneous polynomial of degree j on $\mathbb{R}^d$.

The entire function G, which is a polynomial on $\mathbb{R}^d$, is a polynomial. Finally,

$$(11) \qquad g(z)g(iz) = R(z),$$

where R is a polynomial and the proof of this step is complete.

Step 4. A lemma about entire functions of several variables.

We are now lead to solving the equation (11), where g is an entire function of order 2 of d variables and R is a given polynomial. It is certainly well known that such functions g can be written as $P(z)e^{Q(z)}$, with $Q(z)$ a polynomial of degree at most 2. Moreover, the equation implies that $Q(z) + Q(iz) = 0$, so that Q is homogeneous of degree 2. So we have completed the proof, up to the study

of the equation (11). Since we did not find a simple reference for it, we include the proof of the next lemma, which is a little more general than what we need above.

Lemma 2.3. *Let φ be an entire function of order 2 on $\mathbb{C}^d$ such that, on every complex line, either φ is identically 0 or it has at most N zeros. Then, there exists a polynomial P with degree at most N and a polynomial Q with degree at most 2 such that $\varphi(z) = P(z)e^{Q(z)}$.*

Proof. Without loss of generality, we may assume that $\varphi(0)$ does not vanish. Then, for $z \in \mathbb{C}^d$, $\varphi_z(t) = \varphi(tz)$ is a non-zero entire function of order 2 that has at most N zeros. By Hadamard's factorisation theorem, for every $z \in \mathbb{C}^d$, there exists a polynomial P_z with $\deg(P_z) \leq N$ and $\alpha(z), \beta(z) \in \mathbb{C}$ such that :

$$\varphi(tz) = P_z(t)e^{\alpha(z)t + \beta(z)t^2},$$

with

$$P_z(t) = a_0(z) + \cdots + a_N(z)t^N .$$

From the uniqueness in Hadamard's theorem, we easily see that the functions α and β are homogeneous of degree 1 and 2 respectively, and a_j is of degree j. We may assume that $a_N(z)$ is non identically zero. We have

$$(12) \qquad \varphi(tz)\varphi(-tz)\varphi(itz)\varphi(-itz) = P_z(t)P_z(-t)P_z(it)P_z(-it).$$

Differentiating (12) $4N$ times with respect to t, and then taking $t = 0$, we get that $a_N(z)^4$ is a homogeneous polynomial of degree $4N$. Now,

$$\frac{d}{dt}\big(\varphi(tz)\big)\varphi(-tz)\varphi(itz)\varphi(-itz) = \left(\frac{d}{dt}P_z(t) + (2\beta(z)t + \alpha(z))P_z(t)\right)P_z(it)P_z(-it),$$

and differentiating $4N + 1$ times with respect to t at $t = 0$, we get that $a_N(z)^4\beta(z)$ is holomorphic. Thus $\beta(z)$ is also holomorphic, and so a homogeneous polynomial of degree 2. An analogous proof allows to conclude that $\alpha(z)$ is a homogeneous polynomial of degree 1. Define $Q(z) = \beta(z) + \alpha(z)$ and $P(z) = \varphi(z)e^{-Q(z)}$. We know that P is holomorphic, and we have to prove that it is a polynomial. Then

$$P(tz) = P_z(t) = a_0(z) + \cdots + a_N(z)t^N.$$

In particular,

$$a_j(z) = \frac{1}{j!} \frac{d^j}{dt^j}\big(P(tz)\big)\bigg|_{t=0}$$

is a holomorphic function, thus a homogeneous polynomial of degree j. It follows that $P(z) = a_0(z) + \cdots + a_N(z)$ is a polynomial of degree $\leq N$, which we wanted to prove to conclude for this step. $\qquad\square$

We have also completed the proof of Proposition 2.2. Indeed, g has the required form thus so has f. $\qquad\square$

Let us make a few comments on the proof. Step 3 is very much inspired, with simplifications, from the proof of Hörmander [16]. Step 2 is not contained in [16], and simplifies greatly the proof, even for $N = 0$. Further, it is easy to see that if the function φ in Lemma 2.3 is of order less than k, then $\varphi(z) = P(z)e^{Q(z)}$, with P a polynomial of degree at most N and Q a polynomial of degree at most k.

As we said in the introduction, the reader will find separately the proof of Beurling-Hörmander's Theorem ($N = 0$, $d = 1$) in the Appendix.

3. APPLICATIONS TO OTHER UNCERTAINTY PRINCIPLES.

Let us first mention the following immediate corollary of Theorem 1.1 :

Corollary 3.1. *Let* $f \in L^2(\mathbb{R}^d)$.

  (*i*) *If*

$$(13) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi \sum_{i=1}^d |x_i y_i|} dx dy < +\infty$$

     then $f(x) = P(x)e^{-\sum_{i=1}^d \beta_i x_i^2}$ *with* P *a polynomial and* $\beta_i > 0$ *for* $i = 1, \dots, d$.

  (*ii*) *If*

$$(14) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\widehat{f}(y)|}{(1 + \|x\| + \|y\|)^N} e^{2\pi \|x\| \|y\|} dx dy < +\infty$$

     then $f(x) = P(x)e^{-\beta \|x\|^2}$ *with* P *a polynomial and* $\beta > 0$.

Proof. It is enough to see that Conditions (13) and (14) are stronger than Condition (2) of Theorem 1.1. Thus $f(x) = P(x)e^{-\langle Ax, x \rangle}$ for some positive definite matrix A. A direct computation then shows that the form of the matrix A imposed by Conditions (13) and (14) are respectively A diagonal and $A = \beta I$. □

The next proposition, which implies the Cowling–Price theorem in one dimension, follows at once from the last case.

Proposition 3.2. *Let* $N \geq 0$. *Assume that* $f \in L^2(\mathbb{R}^d)$ *satisfies*

$$\int_{\mathbb{R}^d} \frac{|f(x)|e^{\pi a \|x\|^2}}{(1 + \|x\|)^N} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{|\widehat{f}(y)|e^{\pi b \|x\|^2}}{(1 + \|y\|)^N} dy < +\infty .$$

Then, $ab > 1$ *implies* $f = 0$. *If* $ab = 1$, *then* $f(x) = P(x)e^{-\pi a \|x\|^2}$ *for some polynomial* P.

Another remark is that L^1 norms may be replaced by L^p norms for $1 \leq p \leq \infty$ in the statement of Theorem 1.1. In particular, we get Corollary 1.2 as well as modifications of the usual Hardy's theorem that we state now.

Proposition 3.3 (Hardy type)**.** *Let* $f \in L^2(\mathbb{R}^d)$ *be such that*

$$|f(x)| \leq C(1 + \|x\|)^N e^{-\pi \langle Ax, x \rangle} \quad \text{and} \quad |\widehat{f}(y)| \leq C(1 + \|y\|)^N e^{-\pi \langle By, y \rangle} ,$$

where A *and* B *are two real symmetric positive definite matrices. We have the following implications.*

  1. *If the matrix* $B - A^{-1}$ *is semi-positive definite and non zero, then* $f = 0$.
  2. *If* $B = A^{-1}$, *then* $f(x) = P(x)e^{-\pi \langle Ax, x \rangle}$ *where* P *is a polynomial of degree* $\leq N$.
  3. *Else, there is a dense subspace of functions satisfying these estimates.*

Proof. It is enough to notice that

$$\langle Ax, x \rangle + \langle A^{-1}y, y \rangle \geq 2\langle x, y \rangle$$

and to apply Theorem 1.1 to obtain the second point. The two other cases are easy consequences of case 2. □

Proof of Theorem 1.3. For $d = 1$, this is exactly Proposition 3.2. For $d > 1$, we proceed as in [2] to reduce to the one-dimensional case. For almost every $x' = (x_2, \cdots x_d)$, the function $f_{x'}$ defined by $f_{x'}(x_1) = f(x_1, x')$ is in $L^2(\mathbb{R})$ and has as Fourier transform the function

$$y_1 \mapsto \int_{\mathbb{R}^{d-1}} \widehat{f}(y_1, y')e^{-2\pi i \langle x', y' \rangle} dy' .$$

So, for almost every x', the function $f_{x'}$ satisfies the assumptions of Proposition 3.2, and $e^{\pi a|x_1|^2}f(x_1, x')$ is a polynomial of degree at most $N - 2$ in the x_1 variable. The same is valid in each variable, which allows to conclude. □

Let us remark that, as in [2] for the case $N = 0$, it is possible to weaken the assumption when the conclusion is that f vanishes. We have the immediate corollary:

Corollary 3.4. *Assume that $f \in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)| \frac{e^{\pi a|x_j|^2}}{(1 + |x_j|)^M} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\widehat{f}(y)| \frac{e^{\pi b|y_j|^2}}{(1 + |y_j|)^N} dy < +\infty$$

for some $j = 1, \cdots, d$ and for some positive constants a and b with $ab = 1$. If $\min\{M, N\} = 1$, then f is identically 0.

Proof of Theorem 1.4. It is sufficient to consider the one-dimensional case: for the first statement we conclude the general case from the one-dimensional one as before, and to find a dense subset of functions we use tensorization. So, let us first assume that $f \in L^2(\mathbb{R})$ is such that

$$\int_{\mathbb{R}} |f(x)| e^{2\pi \frac{a^p}{p}|x|^p} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}} |\widehat{f}(y)| e^{2\pi \frac{b^q}{q}|y|^q} dy < +\infty$$

for some positive constants a and b, with f not identically zero. Here $1 < p \leq 2$, and q is the conjugate exponent. It follows from the second inequality that f extends to an entire function, which satisfies the inequality

$$|f(x + iy)| \leq Ce^{\frac{2\pi}{p}\left|\frac{y}{b}\right|^p} .$$

Moreover, the same inequality is valid when f is replaced by its even part, or its odd part. Such a function may be written as $g(z^2)$, or $zg(z^2)$, with g an entire function. One of them is non zero, and satisfies

$$|g(x + iy)| \leq Ce^{2\pi \frac{b-p}{p}|z|^{\frac{p}{2}}} \quad \text{and} \quad \int_{\mathbb{R}} |g(x)| e^{2\pi \frac{a^p}{p}|x|^{\frac{p}{2}}} dx < +\infty .$$

In the second inequality, a has effectively been replaced by an arbitrarily close smaller constant, which we write a again for simplification. We then consider the function

$$G(z) = \int_0^z g(u)du - \int_0^\infty g(u)du .$$

Then G is a non zero entire function of order $\frac{p}{2}$, and, moreover,

$$|G(x + iy)| \leq C|z|e^{2\pi \frac{b-p}{p}|z|^{\frac{p}{2}}} \quad \text{and, for positive } x, \quad |G(x)| \leq Ce^{-2\pi \frac{a^p}{p}|x|^{\frac{p}{2}}} .$$

Let us consider the indicator function of G, given by

$$h_G(\theta) = \limsup_{r \to \infty} \frac{\ln \left| G(re^{i\theta}) \right|}{r^{\frac{p}{2}}}$$

(see [21] for this topic). Then there exists a positive measure on the unit circle, $d\mu$, such that the 2π-periodic function h_G is the convolution of $d\mu$ with the continuous function ϕ_p defined by $\phi_p(\theta) = \cos(\frac{p\theta}{2})$ on $(0, 2\pi)$. An elementary computation shows that

$$\left| \cos(\frac{p\pi}{2}) \right| \phi_p(\theta) + \phi_p(\theta + \pi) \geq 0 .$$

It implies that

$$\left| \cos(\frac{p\pi}{2}) \right| h_G(\theta) + h_G(\theta + \pi) \geq 0 ,$$

and, in particular, $\left|\cos(\frac{p\pi}{2})\right| h_G(\pi) + h_G(0) \geq 0$. But $h_G(\pi) \leq 2\pi\frac{b^{-p}}{p}$, while $h_G(0) \leq -2\pi\frac{a^p}{p}$. The inequality $ab \leq \left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$ follows at once.

Let us now prove that, for $ab < \left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$, there is a dense subset of functions f such that

$$|f(x)| \leq Ce^{-2\pi\frac{a^p}{p}|x|^p} \quad \text{and} \quad |\widehat{f}(y)| \leq Ce^{-2\pi\frac{b^q}{q}|y|^q} .$$

Since this set of functions is stable under multiplications by $e^{2\pi i y_0 x}$ (changing b into an arbitrarily close smaller constant), we see immediately that it is dense, unless it reduces to 0. Indeed, if f is such a non zero function and $\phi \in L^2(\mathbb{R})$ is orthogonal to all functions $e^{2\pi i y_0 x}f$, then $\overline{f}\phi$ is identically 0. Since f is analytic, it means that ϕ is 0.

Moreover, using a dilation if necessary, we can restrict to the case when $a < \left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$ and $b < 1$. We shall prove a slightly stronger result, the fact that E is not reduced to 0, where E is the class of functions for which, for every $a < 1$, there exists a constant C_a such that

$$(15) \qquad |f(x)| \leq C_a e^{-2\pi\frac{a^p}{p}\left|\cos(\frac{p\pi}{2})\right||x|^p} \quad \text{and} \quad |\widehat{f}(y)| \leq C_a e^{-2\pi\frac{a^q}{q}|y|^q} .$$

We claim that this follows easily from the following lemma:

Lemma 3.5. *The function f is in E if and only if it extends to an entire function of order p, and, for every $a < 1$, there exists a constant C_a such that*

$$(16) \qquad |f(x)| \leq C_a e^{-2\pi\frac{a^p}{p}\left|\cos(\frac{p\pi}{2})\right||x|^p} \quad \text{and} \quad |f(iy)| \leq C_a e^{2\pi\frac{a^{-p}}{p}|y|^p} .$$

Indeed, let us take the lemma for granted and finish the proof. We know that there exists non zero entire functions of order p which satisfy these two conditions (see [14] or [21]). So, it follows from the lemma that there exists non zero functions in E.

Proof of the lemma. One implication (not the one that we need) is trivial. So let us assume that f is an entire function which satisfies (16). We adapt the proof from [14] with the required modifications to have a precise control of the constants. We first give precise estimates of the function inside each of the quadrants of the complex plane which are delimited by the coordinate axes. It follows from Phragmèn-Lindelöf principle and the assumption that, for every $\varepsilon > 0$ there exists a constant C_ε such that $\theta \in [0, \pi/2]$

$$\left|f(re^{i\theta})\right| \leq C_\varepsilon e^{\frac{2\pi}{p}\cos(p(\theta-\frac{\pi}{2}))r^p} e^{\varepsilon r^p} .$$

From the convexity of the function $\theta \mapsto -\log(\cos\theta)$, we see that $\cos(\frac{p(\theta-\pi)}{2}) \leq (\sin\theta)^p$. Combining the corresponding inequality for f with a simple estimate of f in an angular sector around the x-axis, and doing the same in the three other quadrants, we deduce the existence of constants $\alpha > 1$ and $\beta > 0$ such that, for every $a > 1$, there exists a constant C_a, such that,

$$(17) \qquad |f(z)| \leq C_a e^{a\frac{2\pi}{p}|y|^p} \quad \text{and, for } |x| \geq \alpha|y|, \quad |f(z)| \leq C_a e^{-\beta|x|^p} .$$

Let us now deduce from these last inequalities the required exponential decrease of the Fourier transform of f. It is sufficient to prove that, for every $a > 1$, there exists a contant C_a such that, for all positive integers n and for all $y \in \mathbb{R}$,

$$\left(\frac{2\pi}{q}\right)^n |y|^{nq}\left|\widehat{f}(y)\right| \leq C_a a^n \Gamma(n+1) .$$

This is equivalent to proving (possibly by changing the constant C_a), that, for all positive integers n and for all $y \in \mathbb{R}$,

$$|y|^n\left|\widehat{f}(y)\right| \leq C_a a^n \left(\frac{q}{2\pi}\right)^{\frac{n}{q}} \Gamma(\frac{n}{q}+1) .$$

This in turn is implied by the inequality

$$\int_{\mathbb{R}} \left| f^{(n)}(x) \right| dx \le C_a a^n q^{\frac{n}{q}} \Gamma\left(\frac{n}{q} + 1 \right) (2\pi)^{\frac{n}{p}} ,$$

which we now prove using (17). To estimate $\left| f^{(n)}(x) \right|$, we use the Cauchy inequality related to the circle centered at x and with radius $R = \left(\frac{n}{2\pi} \right)^{\frac{1}{p}}$. For $|x| \le 2\alpha R$, we use the first part of (17), to get the inequality

$$\left| f^{(n)}(x) \right| \le C_a \frac{\Gamma(n+1)}{R^n} e^{a\frac{2\pi}{p}R^p} \le C_a' a^n (2\pi)^{\frac{n}{p}} e^{\frac{n}{p}} \frac{\Gamma(n+1)}{n^{\frac{n}{p}}} .$$

Then Stirling's formula for the asymptotics of Γ allows to conclude for the integral on the interval $|x| \le 2\alpha R$. Outside, we get the estimate

$$\left| f^{(n)}(x) \right| \le C_a (2\pi)^{\frac{n}{p}} \frac{\Gamma(n+1)}{n^{\frac{n}{p}}} e^{-\beta'|x|^p} .$$

We conclude as before using the Stirling's formula.   □

This finishes the proof of Theorem 1.4.   □

It leaves completely open the study of the critical case, $ab = \left| \cos(\frac{p\pi}{2}) \right|^{\frac{1}{p}}$, except for the case when $p = 2$, for which the previous theorems give precise informations.

4. Properties of the ambiguity function

For sake of self-containedness, let us recall here a few properties of the ambiguity function that we may use in the sequel. They can be found in [1], [24].

Lemma 4.1. *Let u, v in $L^2(\mathbb{R}^d)$. For $a, \omega \in \mathbb{R}^d$, $\lambda > 0$ define*

$$S(a)u(t) = u(t-a) \quad , \quad M(\omega)u(t) = e^{2i\pi\omega t}u(t) \quad , \quad D_\lambda u(t) = \lambda^{\frac{d}{2}} u(\lambda t)$$

and recall that we defined $Zu(t) = u(-t)$. Then

  (i) $A\big(S(a)u, S(b)v\big)(x,y) = e^{i\pi\langle a+b, y \rangle} A(u,v)(x+b-a, y),$
  (ii) $A\big(M(\omega_1)u, M(\omega_2)v\big)(x,y) = e^{i\pi\langle \omega_1+\omega_2, x \rangle} A(u,v)(x, y+\omega_1-\omega_2),$
  (iii) $A\big(D_\lambda u, D_\lambda v\big)(x,y) = A(u,v)\left(\lambda x, \frac{y}{\lambda} \right),$
  (iv) $A(Zu, Zv)(x,y) = \overline{A(u,v)(x,y)},$
  (v) $A(\widehat{u}, \widehat{v})(x,y) = A(u,v)(y, -x)$ *and* $A(u,v)(x,y) = \overline{A(v,u)(-x,-y)}.$

Lemma 4.2. *Let $u, v \in L^2(\mathbb{R}^d)$. Then $A(u,v)$ is continuous on $\mathbb{R}^{2d}$ and $A(u,v) \in L^2(\mathbb{R}^{2d})$. Further,*

$$\|A(u,v)\|_{L^2(\mathbb{R}^{2d})} = \|u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^2(\mathbb{R}^d)}$$

Proof. This fact is also well known, however to help the reader get familiar with our notation, let us recall the proof of the last assertion (see [1], [24]). If u and v are fixed, we will write

(18) $$h_x(t) = u(t + \frac{x}{2})\overline{v(t - \frac{x}{2})}$$

The change of variables

(19) $$\xi = t - \frac{x}{2} \quad \text{and} \quad \eta = t + \frac{x}{2}$$

gives

$$\iint |h_x(t)|^2 dt dx = \iint |u(\eta)|^2 |v(\xi)|^2 d\eta d\xi = \|u\|^2_{L^2(\mathbb{R}^d)} \|v\|^2_{L^2(\mathbb{R}^d)}.$$

In particular, for almost every x, the integral with respect to t is finite, $i.e$ $h_x \in L^2$. Noticing that $A(u)(x, y) = \widehat{h_x}(y)$, and using Parseval's formula we obtain

$$\int \left(\int |A(u, v)(x, y)|^2 dx \right) dy = \int \left(\int \left| \widehat{h_x}(y) \right|^2 dy \right) dx = \|u\|_{L^2(\mathbb{R}^d)}^2 \|v\|_{L^2(\mathbb{R}^d)}^2,$$

which completes the proof. $\qquad\square$

Finally, we will also need the following lemma from [17], [20]:

Lemma 4.3. *Let* $u, v, w \in L^2(\mathbb{R}^d)$. *Then, for every* $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^{2d}} A(u, v)(s, t) \overline{A(v, w)(s, t)} e^{2i\pi(\langle s, x \rangle + \langle t, y \rangle)} ds dt = A(u, v)(-y, x) \overline{A(v, w)(-y, x)}.$$

5. Heisenberg inequality for the ambiguity function.

We show here that the ambiguity function is subject to sharp inequalities of Heisenberg type. We give first a directional version of Heisenberg's inequality (1) in the context of ambiguity functions, with an elementary proof.

Theorem 5.1. *For* $u, v \in L^2(\mathbb{R}^d)$, *for every* $i = 1, \dots, d$ *and every* $a, b \in \mathbb{R}$, *one has the following inequality:*

$$(20) \quad \iint_{\mathbb{R}^{2d}} |x_i - a|^2 |A(u, v)(x, y)|^2 dx dy \iint_{\mathbb{R}^{2d}} |y_i - b|^2 |A(u, v)(x, y)|^2 dx dy \geq \frac{\|u\|_{L^2(\mathbb{R}^d)}^4 \|v\|_{L^2(\mathbb{R}^d)}^4}{4\pi^2}.$$

Moreover equality holds in (20), with u and v non identically 0, if and only if there exists $\mu, \nu \in L^2(\mathbb{R}^{d-1})$, $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$ such that

$$u(t) = \mu(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_d) e^{2i\pi\beta t_i} e^{-\alpha/2|t_i - \gamma|^2},$$
$$v(t) = \nu(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_d) e^{2i\pi(b+\beta)t_i} e^{-\alpha/2|t_i - a - \gamma|^2}.$$

To prove the theorem, we will need the following lemma that has its own interest:

Lemma 5.2. *Let* $u, v \in L^2(\mathbb{R}^d)$ *be both non identically zero and let* $i = 1, \dots, d$. *The following are equivalent:*

(a) $\iint |x_i|^2 |A(u, v)(x, y)|^2 dx dy < +\infty.$

(b) For all $a \in \mathbb{R}$, $\iint |x_i - a|^2 |A(u, v)(x, y)|^2 dx dy < +\infty.$

(c) $\int |t_i|^2 |u(t)|^2 dt < +\infty$ and $\int |t_i|^2 |v(t)|^2 dt < +\infty.$

(d) For all $a, b \in \mathbb{R}$, $\int |t_i - a|^2 |u(t)|^2 dt < +\infty$ and $\int |t_i - b|^2 |v(t)|^2 dt < +\infty.$

Remark: Note that, if $u \in L^2(\mathbb{R}^d)$ and if $\int |t_i|^2 |u(t)|^2 dt < +\infty$ then $t_i |u(t)|^2 \in L^1(\mathbb{R}^d)$.

Proof. Let us first remark that $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$. Indeed, it is sufficient to use the triangle inequality

$$|x|^2 \leq 2(|x - a|^2 + |a|^2).$$

Moreover, (b) and (d) may be replaced by (b') and (d'), where "*for all*" has been changed into "*for some*". Let us prove that (a) implies (d'). With the notations of Lemma 4.2, Parseval identity gives:

$$\iint |x_i|^2 |A(u,v)(x,y)|^2 dxdy = \int |x_i|^2 \int \left|\widehat{h_x}(y)\right|^2 dydx$$

$$= \int |x_i|^2 \int |h_x(t)|^2 dtdx$$

$$= \iint |x_i|^2 \left|u(t+\frac{x}{2})\overline{v(t-\frac{x}{2})}\right|^2 dtdx$$

$$(21) \qquad = \int \left(\int |\eta_i - \xi_i|^2 |v(\xi)|^2 d\xi\right) |u(\eta)|^2 d\eta.$$

So, if (a) holds, for almost every η, $|u(\eta)|^2 \int |\xi_i - \zeta_i|^2 |v(\xi)|^2 d\xi < +\infty$. As we assumed that $u \neq 0$, there exists η such that $u(\eta) \neq 0$, and the first inequality in (d') holds with $a = \eta$. Since u and v play the same role, we conclude for the second part similarly.

Conversely, if (c) holds, the right hand side of (21) is finite, and (a) holds also. $\qquad \square$

Proof of theorem 5.1. Let us start by proving the inequality. Set $A = A(u,v)$. We may assume that

$$\left[\iint |x_i|^2 |A(x,y)|^2 dxdy\right]\left[\iint |y_i|^2 |A(x,y)|^2 dxdy\right] < +\infty$$

so that both factors are finite, and, by homogeneity, that $\|u\|_{L^2(\mathbb{R}^d)} = \|v\|_{L^2(\mathbb{R}^d)} = 1$. Moreover, replacing u and v by translates of these functions (with the same translation), we may assume that

$$(22) \qquad -\int t_i |u(t)|^2 dt = \int t_i |v(t)|^2 dt = \frac{\overline{a}}{2}.$$

Heisenberg's inequality (1) applied to $h_x(t) = u\left(t + \frac{x}{2}\right)\overline{v\left(t - \frac{x}{2}\right)}$ implies that, for any $b \in \mathbb{R}^d$:

$$(23) \qquad \frac{1}{4\pi}\int |h_x(t)|^2 dt \leq \left(\int |t_i|^2 |h_x(t)|^2 dt\right)^{1/2}\left(\int |y_i - b|^2 |A(u,v)(x,y)|^2 dy\right)^{1/2}.$$

Integrating this inequality with respect to the x-variable and appealing to Cauchy-Schwarz' inequality, we get:

$$(24) \qquad \frac{1}{4\pi} = \frac{\|u\|^2\|v\|^2}{4\pi} \leq \left(\iint |t_i|^2 |h_x(t)|^2 dtdx\right)^{\frac{1}{2}}\left(\iint |y_i - b|^2 |A(u,v)(x,y)|^2 dxdy\right)^{\frac{1}{2}}.$$

Let us now transform the first factor on the right hand side of this expression. We write

$$|t_i|^2 = \left(t_i + \frac{x_i - a}{2}\right)\left(t_i - \frac{x_i - a}{2}\right) + \frac{|x_i - a|^2}{4}.$$

The second term which appears is, by Parseval identity, equal to

$$\frac{1}{4}\iint |x_i - a|^2 |A(x,y)|^2 dxdy\,;$$

The first term is equal to

$$(25) \qquad \iint \left(\eta_i - \frac{a}{2}\right)\left(\xi_i + \frac{a}{2}\right)|u(\eta)|^2 |v(\xi)|^2 d\eta d\xi = -\frac{1}{4}|\overline{a} - a|^2 \leq 0$$

using (22). Finally, including these results in (24), we obtain the desired inequality.

Assume now that we have equality. Let us remark that, using properties (a) and (b) of Lemma 4.1, we may as well assume that the constants a and b are 0. Moreover, up to a same translations in space and frequency, we may again assume that $-\int t_i |u(t)|^2 dt = \int t_i |v(t)|^2 dt = \frac{\overline{a}}{2}$, and $-\int t_i |\mathcal{F}u(t)|^2 dt =$

$\int t_i |\mathcal{F}v(t)|^2 dt = \frac{\overline{b}}{2}$. Let $h_x(t) = u\left(t + \frac{x}{2}\right) \overline{v\left(t - \frac{x}{2}\right)}$ as before. Then, to have equality in (20), we have equality in (25), *i.e.* we have $\overline{a} = a = 0$. Similarly, exchanging the roles of the x and y variables, we also have $\overline{b} = b = 0$.

We then have equality in Cauchy-Schwarz's inequality (24). This implies that

$$x \mapsto \int |t_i|^2 |h_x(t)|^2 dt \text{ and } x \mapsto \int |y_i|^2 \left|\widehat{h_x}(y)\right|^2 dy$$

are proportional.

Further, for almost every x, we also have equality in Heisenberg's inequality (23). From now on, we assume for simplicity that $i = 1$. We then get that, for almost every x, h_x is a Gaussian in the t_1 variable:

$$h_x(t) = C(x, t')e^{-\pi\alpha(x)|t_1|^2}.$$

where $t' = (t_2, \ldots, t_d) \in \mathbb{R}^{d-1}$, $C(x, t')$ is not identically 0 and $\alpha(x) > 0$ for those x for which $C(x, t') \neq 0$.

Let us first prove that α does not depend on x. With this expression of h_x, we get that

$$\int |t_1|^2 |h_x(t)|^2 dt = \|C(x, .)\|_{L^2(\mathbb{R}^{d-1})}^2 \frac{\sqrt{\pi}}{2}\left(\frac{1}{2\pi\alpha(x)}\right)^{\frac{3}{2}}$$

whereas (with Parseval identity)

$$\int |y_i|^2 \left|\widehat{h_x}(y)\right|^2 dy = \|C(x, .)\|_{L^2(\mathbb{R}^{d-1})}^2 \frac{1}{4\pi}\sqrt{\frac{\alpha(x)}{2}}.$$

But these two functions are proportional only if $\alpha(x)$ is a constant, say $\alpha(x) = \alpha$.

Taking inverse Fourier transforms, we get that

$$(26) \qquad\qquad A(u, v)(x, y) = \frac{1}{\sqrt{\alpha}}\widehat{C}(x, -\widehat{y}')e^{-\frac{\pi}{\alpha}|y_1|^2},$$

where $\widehat{C}(x, .)$ is the Fourier transform in $\mathbb{R}^{d-1}$ of $C(x, .)$. In particular, as $A(u, v)$ is continuous, $\widehat{C}$ is also continuous.

Further, if one has equality in (20) for u, v, one has again this equality if u, v are replaced by their Fourier transform, as $A(\widehat{u}, \widehat{v})(x, y) = A(u, v)(y, -x)$. So, we have

$$(27) \qquad\qquad A(u, v)(x, y) = D(y, x')e^{-\frac{\pi}{\beta}|x_1|^2},$$

for some function D and some $\beta > 0$. Comparing (26) and (27), we get that

$$A(u, v)(x, y) = E(x', y')e^{-\frac{\pi}{\beta}|x_1|^2}e^{-\frac{\pi}{\alpha}|y_1|^2}$$

for some function $E \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$. Taking the Fourier transform in the y variable, we get

$$u\left(t + \frac{x}{2}\right)\overline{v\left(t - \frac{x}{2}\right)} = \widetilde{E}(x', t')e^{-\frac{\pi}{\beta}|x_1|^2}e^{-\pi\alpha|t_1|^2}.$$

So, setting again $\eta = t + \frac{x}{2}$, $\xi = t - \frac{x}{2}$, we get

$$u(\eta)\overline{v(\xi)} = \widetilde{E}\left(\eta' - \xi', \frac{\eta' + \xi'}{2}\right)e^{-\frac{\pi}{\beta}(\eta_1 - \xi_1)^2}e^{-\frac{\pi\alpha}{4}(\eta_1 + \xi_1)^2}.$$

This is only possible if $\beta = \frac{4}{\alpha}$ and $u(x) = \mu(\widehat{x}_i)e^{-\frac{\pi\alpha}{2}|x_i|^2}$, $v(x) = \nu(\widehat{x}_i)e^{-\frac{\pi\alpha}{2}|x_i|^2}$ with $\mu, \nu \in L^2(\mathbb{R}^{d-1})$. $\qquad\square$

We now translate the last theorem in terms of inequalities on covariance matrices, as it is classical for inequality (1) (see for instance [9]).

Proof of Theorem 1.5. First, it follows from Lemma 5.2 that X and Y have moments of order 2. Let us prove that X and Y are not correlated. Without loss of generality, we may assume that

$$\int t_i |u(t)|^2 dt = \int t_i |v(t)|^2 dt = \int t_i |\mathcal{F}u(t)|^2 dt = \int t_i |\mathcal{F}v(t)|^2 dt = 0$$

for all i, so that X and Y are centered. Let us show that

$$\iint x_i y_j |A(u,v)|^2 dx\, dy = 0 .$$

Using Plancherel identity , we are led to consider

$$\iint x_i \left[\partial_{t_j} h_x(t) \overline{h_x(t)} + h_x(t) \overline{\partial_{t_j} h_x(t)} \right] dxdt .$$

Writing $x_i = t + \frac{x_i}{2} - (t - \frac{x_i}{2})$, and

$$\partial_{t_j} h_x(t) = u(t + \frac{x}{2}) \overline{\partial_{t_j} v(t - \frac{x}{2})} + \partial_{t_j} u(t + \frac{x}{2}) \overline{v(t - \frac{x}{2})},$$

we get eight terms. For four of them we get directly 0. The sum of the last four may be written, after changes of variables, as

$$\int_{\mathbb{R}^d} t_i \partial_{t_j} [|u(t)|^2] dt \times \int_{\mathbb{R}^d} |v(t)|^2 dt - \int_{\mathbb{R}^d} |u(t)|^2 dt \times \int_{\mathbb{R}^d} t_i \partial_{t_j} [|v(t)|^2] dt .$$

After an integration by parts, the two terms give δ_{ij}, so that their difference is 0.

Let us now prove the second assertion. Let C be an automorphism of $\mathbb{R}^d$. For a function $f \in L^2(\mathbb{R}^d)$, we consider f_C the function given by $f_C(t) = |\det(C)|^{-\frac{1}{2}} f(C^{-1}t)$. Then a simple change of variables shows that the probability density of $(CX, {}^t C^{-1} Y)$ is $|A(u_C, v_C)|^2$. Eventually changing u and v into u_C and v_C we may assume that $V(Y)$ is the identity matrix. Moreover, we may also assume that $V(X)$ is diagonal. Then the inequality follows from Theorem 5.1. Equality holds only if all eigenvalues are equal to $4\pi^2$, which means that u and v are Gaussian functions. □

From Theorem 1.5, we immediately obtain that

$$(28) \qquad\qquad \det(V(X)) \times \det(V(Y)) \geq (4\pi^2)^{-2d} .$$

Equality holds only for Gaussian functions. Another (much less elementary) proof of (28) can be obtained using the entropy inequality of Lieb and the theorem of Shannon (see [11], Section 6).

The same inequality holds for traces instead of determinants. We state it independently.

Corollary 5.3. *Let $u, v \in L^2(\mathbb{R}^d)$ be both non identically zero and $a, b \in \mathbb{R}^d$. Then*

$$\iint_{\mathbb{R}^{2d}} \|x - a\|^2 |A(u,v)(x,y)|^2 dxdy \times \iint_{\mathbb{R}^{2d}} \|y - b\|^2 |A(u,v)(x,y)|^2 dxdy$$

$$(29) \qquad\qquad\qquad \geq \frac{d^2 \|u\|^4_{L^2(\mathbb{R}^d)} \|v\|^4_{L^2(\mathbb{R}^d)}}{4\pi^2},$$

with equality if and only if there exists $\lambda, \nu \in \mathbb{C}^*$, $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}^d$ such that

$$u(t) = \lambda e^{2i\pi \langle \beta, t \rangle} e^{-\alpha/2 \|t - \gamma\|^2} \quad \text{and} \quad v(t) = \nu e^{2i\pi \langle b + \beta, t \rangle} e^{-\alpha/2 \|t - a - \gamma\|^2} .$$

6. UNCERTAINTY PRINCIPLES FOR THE AMBIGUITY FUNCTION.

We first prove the following uncertainty principle for the ambiguity function, which also gives a characterization of Hermite functions:

Theorem 6.1. *Let $u, v \in L^2(\mathbb{R}^d)$ be non identically vanishing. If*

$$(30) \qquad \iint_{\mathbb{R}^{2d}} |A(u,v)|^2 \frac{e^{\pi \|x\|^2}}{(1+\|x\|)^N} dxdy < +\infty \quad \text{and} \quad \iint_{\mathbb{R}^{2d}} |A(u,v)|^2 \frac{e^{\pi \|y\|^2}}{(1+\|y\|)^N} dxdy < +\infty$$

for all $j = 1, \cdots, d$, then there exists $a, w \in \mathbb{R}^d$ such that both u and v are of the form

$$P(x)e^{2i\pi \langle w,x \rangle} e^{-\pi \|x-a\|^2},$$

with P a polynomial.

Let us first remark that when $u = v$, since the Fourier transform of $|A(u,u)|^2$ taken at (x,y) is equal to $|A(u,u)|^2(y,-x)$, the result follows immediately from Theorem 1.3. Our aim is to prove that it is also valid in the general case. We first prove a weaker statement.

Proposition 6.2. *Let $u, v \in L^2(\mathbb{R}^d)$ be non identically vanishing. If*

$$(31) \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|A(u,v)(x,y)|^2}{(1+\|x\|+\|y\|)^N} e^{\pi(\|x\|^2+\|y\|^2)} dxdy < +\infty,$$

then there exists $a, w \in \mathbb{R}^d$ such that both u and v are of the form $P(x)e^{2i\pi \langle w,x \rangle} e^{-\pi \|x-a\|^2}$ with P a polynomial.

Before starting the proof of Proposition 6.2, let us state two lemmas. The first one is elementary and well known.

Lemma 6.3. *Let $u, v \in L^2(\mathbb{R}^d)$ be non identically vanishing. Then*

$$u(x) = P(x)e^{2i\pi \langle \alpha,x \rangle} e^{-\pi \|x-a\|^2} \text{ and } v(x) = Q(x)e^{2i\pi \langle \alpha,x \rangle} e^{-\pi \|x-a\|^2}$$

with P, Q polynomials and $a, \alpha \in \mathbb{R}^d$ if and only if there is a polynomial R such that

$$A(u,v)(x,y) = R(x,y)e^{2i\pi(\langle \alpha,x \rangle + \langle a,y \rangle)} e^{-\frac{\pi}{2}(\|x\|^2+\|y\|^2)}.$$

Moreover, $\deg(R) = \deg(P) + \deg(Q)$.

Lemma 6.4. *Assume that $u, v \in L^2(\mathbb{R}^d)$, with $u(x) = P(x)e^{2i\pi \langle a,x \rangle} e^{-\pi \|x\|^2}$, where $a \in \mathbb{C}^d$. Then the function $A(u,v)$ can be extended to an entire function on $\mathbb{C}^{2d}$.*

Proof. For $z, \zeta \in \mathbb{C}^d$, we note $\langle z, \zeta \rangle = \sum z_i \zeta_i$. Then

$$(32) \qquad A(u,v)(x,y) = e^{i\pi \langle x,(y+2a) \rangle} \times \int P(t+x)\overline{v(t)}e^{-\pi\|t+x\|^2} e^{2i\pi \langle t,(a+y) \rangle} dt.$$

This clearly makes sense for $x, y \in \mathbb{C}^d$, and defines an entire function. $\square$

Proof of Proposition 6.2 . By homogeneity, we may assume that $\|u\|_{L^2} = \|v\|_{L^2} = 1$. For each $w \in L^2(\mathbb{R}^d)$, we consider the function

$$g_w = A(u,v)\overline{A(v,w)}.$$

By Lemma 4.3, the Fourier transform of g_w is given by $\mathcal{F}g_w(x,y) = g_w(y,-x)$.

First step. There exists a polynomial P such that

$$(33) \qquad A(u,v)(x,y)\overline{A(v,u)}(x,y) = P(x,y)e^{-\pi(\|x\|^2+\|y\|^2)}.$$

Proof. We consider here the function g_u. As g_u is (up to a change of variable) its own Fourier transform, by Proposition 3.2, it is sufficient to prove that

$$(34) \qquad \int |g_u(x,y)| \frac{e^{\pi(\|x\|^2 + \|y\|^2)}}{(1 + \|x\| + \|y\|)^N} dx dy < \infty .$$

It follows immediately from the assumption on $A(u,v)$, using Cauchy-Schwarz inequality and the fact that $\overline{A(v,u)(x,y)} = A(u,v)(-x,-y)$. □

To complete the proof of the proposition, it is sufficient to prove that $A(u,v)$ extends to an entire function of order 2. Indeed, Lemma 2.3 then implies that

$$A(u,v)(x,y) = R(x,y)e^{Q(x,y)},$$

where R is a polynomial and Q a polynomial of degree at most 2. But, as

$$A(u,v)(x,y)A(u,v)(-x,-y) = A(u,v)(x,y)\overline{A(v,u)(x,y)} = P(x,y)e^{-\pi(\|x\|^2 + \|y\|^2)},$$

we get $Q(x,y) = \langle \beta, x \rangle + \langle \gamma, y \rangle - \frac{\pi}{2}(\|x\|^2 + \|y\|^2)$ for some constants $\beta, \gamma \in \mathbb{C}^d$. Next, Condition (31) implies that β, γ are purely imaginary, $\beta = 2i\pi\alpha, \gamma = 2i\pi a$ with $a, \alpha \in \mathbb{R}^d$ so that

$$A(u,v)(x,y) = R(x,y)e^{2i\pi(\langle \alpha, x \rangle + \langle a, y \rangle)}e^{-\frac{\pi}{2}(\|x\|^2 + \|y\|^2)},$$

with R a polynomial. Lemma 6.3 allows to conclude. So, we have finished the proof once we have proved the second step.

Second step. The function $A(u,v)$ extends to an entire function of order 2.

Proof. To prove this, we first show that, for each $w \in L^2(\mathbb{R}^d)$, the function g_w extends to an entire function of order 2. Since, up to a change of variable, g_w coincides with its Fourier transform, it is sufficient to show that

$$\int |g_w(x,y)|e^{\frac{\pi}{4}(\|x\|^2 + \|y\|^2)} dx dy < \infty .$$

This last inequality follows from the fact that $|g_w| \leq |A(u,v)|\|w\|_{L^2}$, and from the assumption on $A(u,v)$. We get the estimate

$$(35) \qquad |g_w(z,\zeta)| \leq C\|w\|e^{4\pi(\|z\|^2 + \|\zeta\|^2)},$$

for all $(z,\zeta) \in \mathbb{C}^d \times \mathbb{C}^d$, with a constant C which does not depend on w. In order to conclude this step, it is sufficient to show that there exists a constant C such that, for each $(z,\zeta) \in \mathbb{C}^d \times \mathbb{C}^d$, we may find $w_{z,\zeta}$ which is of the form required in Lemma 6.4, such that

$$(36) \qquad |A(w_{z,\zeta},v)(z,\zeta)| \geq C^{-1}e^{-C(\|z\|^2 + \|\zeta\|^2)},$$

and

$$(37) \qquad \|w_{z,\zeta}\| \leq C^{-1}e^{C(\|z\|^2 + \|\zeta\|^2)}.$$

By density of the Hermite functions we can choose a polynomial P_0 such that

$$\int P_0(t)\overline{v(t)}e^{-\pi\|t\|^2} dt = 1 .$$

We then define $w_{z,\zeta}$ by

$$w_{z,\zeta}(t) = P_0(t-z)e^{2\pi\langle z - i\zeta, t \rangle}e^{-\pi\|t\|^2} .$$

It follows from the choice of P_0 that

$$A(w_{z,\zeta},v)(z,\zeta) = e^{\pi(\langle z,z \rangle - i\langle z,\zeta \rangle)} .$$

Then (36) and (37) follow from direct computations. Finally, since $A(u,v)\overline{A(v,w_{z,\zeta})}$ extends to an entire function for each z,ζ, and since the second factor is also entire and does not vanish in a

neighborhood of $(-z, -\zeta) \in \mathbb{C}^d$, $A(u, v)$ extends also to an entire function. The fact that it is of order 2 follows from (35), (36) and (37). $\qquad\square$

We have completed the proof of Proposition 6.2. $\qquad\square$

Proof of Theorem 6.1. With the weaker assumption (30), we conclude that (33)also holds, using the directional theorem for Fourier transforms. We claim that $A(u, v)$ is an analytic function of each of the variables x and y. Indeed, as before, for every function $w \in L^2(\mathbb{R}^d)$, the product $A(u, v)\overline{A(v, w)}$ extends to a holomorphic function of x, y being fixed, as well as to a holomorphic function of y, x being fixed. When choosing w as before, we conclude that the function extends to an entire function of order 2 in x, for fixed y. So, for almost every fixed y, $A(u, v)(x, y)$ may be written as $R_y(x)e^{2i\pi\langle\alpha(y),x\rangle}e^{-\frac{\pi}{2}\|x\|^2}$, with R_y a polynomial of degree at most N. It follows that the continuous function

$$e^{\pi(\|x\|^2+\|y\|^2)}|A(u, v)(x, y)|^2$$

is a polynomial of degree at most $2N$ in each variable x, y. So it is a polynomial, and the assumption (31) is satisfied. We can now use Proposition 6.2 to conclude. $\qquad\square$

In particular, it follows from Theorem 6.1 that there does not exist two non zero functions u, v which satisfy Condition (30) for $N \leq d$. In this case, where the conclusion is that u or v is identically 0, the condition can be relaxed into a directional condition as for the case of the Fourier transform.

Corollary 6.5. *Assume that*

$$(38) \qquad \iint_{\mathbb{R}^{2d}} |A(u, v)|^2 \frac{e^{\pi|x_j|^2}}{(1+|x_j|)^M} dxdy < +\infty \quad \text{and} \quad \iint_{\mathbb{R}^{2d}} |A(u, v)|^2 \frac{e^{\pi|y_j|^2}}{(1+|y_j|)^N} dxdy < +\infty$$

for some $j = 1, \cdots, d$. If $\min\{M, N\} = 1$, then u or v vanishes.

Proof. When $d = 1$, the result follows from Theorem 6.1. Let us now consider the case $d > 1$. We assume that the condition is satisfied for $j = 1$. For $t' \in \mathbb{R}^{d-1}$ and for f in $L^2(\mathbb{R}^d)$, we define $f_{t'}(t_1) = f(t_1, t')$. It follows from Plancherel identity that

$$\int_{\mathbb{R}^{d-1}} |A(u, v)(x, y)|^2 dy' = \int_{\mathbb{R}^{d-1}} \left|A(u_{t'-\frac{x'}{2}}, v_{t'+\frac{x'}{2}})(x_1, y_1)\right|^2 dy' .$$

Changing variables, and using Fubini's Theorem, it follows that, for almost every ξ' and η' in $\mathbb{R}^{d-1}$,

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{|A(u_{\eta'}, v_{\xi'})(x_1, y_1)|^2}{(1+|x_1|+|y_1|)^2} e^{\pi(|x_1|^2+|y_1|^2)} dx_1 dy_1 < +\infty .$$

It follows from the one dimensional case that either $u_{\eta'}$ or $v_{\xi'}$ is identically 0. So, for almost every ξ' and η' in $\mathbb{R}^{d-1}$, $A(u_{\eta'}, v_{\xi'}) = 0$. It follows that $A(u, v) = 0$. $\qquad\square$

As in section 3, we may deduce from Theorem 6.2 a version of Hardy's theorem for the ambiguity function. We state it here in a larger context, allowing general positive definite quadratic forms. We do not give the proof, which is obtained from a small modification of the proof of Theorem 6.2. Let us remark that the constraints on degrees are always elementary.

Corollary 6.6. *Let $u, v \in L^2(\mathbb{R}^d)$ and assume that*

$$|A(u, v)(x, y)| \leq (1 + \|x\|)^N (1 + \|y\|)^N e^{-\frac{1}{2}\langle Cx,x\rangle - \frac{1}{2}\langle By,y\rangle} .$$

We have the following implications.

1. *If $B - \pi^2 C^{-1}$ is positive, non zero, then either u or $v = 0$.*
2. *If $B = \pi^2 C^{-1}$, there are polynomials P, Q of degree $\leq N$ and constants $\omega, a \in \mathbb{R}^d$ such that*

$$u(x) = P(x)e^{2i\pi\langle\omega,x\rangle}e^{-\langle C(x-a),x-a\rangle} \text{ and } v(x) = Q(x)e^{2i\pi\langle\omega,x\rangle}e^{-\langle C(x-a),x-a\rangle} .$$

Remark: This corollary implies in particular that if $u_0(x) = P_0(x)e^{-\alpha\|x\|^2}$, $v_0(x) = Q_0(x)e^{-\alpha\|x\|^2}$ with P_0, Q_0 polynomials and if $|A(u,v)| = |A(u_0,v_0)|$ then u, v are of the form

$$u(x) = P(x)e^{2i\pi\langle\omega,x\rangle - \alpha\|x-a\|^2}$$
$$v(x) = Q(x)e^{2i\pi\langle\omega,x\rangle - \alpha\|x-a\|^2}$$

with P, Q polynomials and $\omega, a \in \mathbb{R}^d$.

The problem of finding u, v from u_0, v_0 is known as the *radar ambiguity problem* and has been considered by Bueckner [5] and de Buda [4] for u_0, v_0 as above. This remark corrects the proof of [4].

Further references on this problem may be found in [18] and [12].

Let us finally give a Gel'fand–Shilov type theorem.

Theorem 6.7. *Let $1 < p < 2$, and let q be the conjugate exponent. Assume that $u, v \in L^2(\mathbb{R}^d)$ satisfy*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |A(u,v)(x,y)|^2 e^{2\pi\frac{a^p}{p}|x_j|^p} dxdy < +\infty \quad \text{and} \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} |A(u,v)(x,y)|^2 e^{2\pi\frac{b^q}{q}|y_j|^q} dxdy < +\infty$$

for some $j = 1, \cdots, d$ and for some positive constants a and b. Then either u or v vanish if $ab > \left|\cos\left(\frac{p\pi}{2}\right)\right|^{\frac{1}{p}}$. If $ab < \left|\cos\left(\frac{p\pi}{2}\right)\right|^{\frac{1}{p}}$, there exists a non zero function u such that the two conditions are satisfied by $A(u,u)$.

Proof. It is sufficient to consider the one dimensional case. Otherwise, the proof works as in Corollary 6.5. Let us prove the first assertion. If we proceed as in the proof of Theorem 6.1, we conclude at once that

$$A(u,v)(x,y)A(u,v)(-x,-y) = 0$$

, using the similar result on Fourier transforms. It remains to show that $A(u,v)$ is an analytic function of each variable, which is obtained in the same way as before using an auxiliary function w.

Let us now prove that for $ab < \left|\cos\left(\frac{p\pi}{2}\right)\right|^{\frac{1}{p}}$, there exists a non zero function u such that the two conditions are satisfied by $A(u,u)$. Using Plancherel formula, the first condition may as well be written as

$$\iint_{\mathbb{R}\times\mathbb{R}} |u\left(t - \frac{x}{2}\right)|^2 |u\left(t + \frac{x}{2}\right)|^2 e^{2\pi\frac{a^p}{p}|x|^p} dxdy < +\infty .$$

Using the same change of variable as before, and the inequality

$$|\eta - \xi| \le 2^{p-1}(|\eta|^p + |\xi|^p),$$

we see that this integral is bounded by the square of the integral

$$\int_{\mathbb{R}} |u(\xi)|^2 e^{4\pi\frac{(2^{1-2/p}a)^p}{p}|\xi|^p} d\xi .$$

For the second integral, we use Lemma 4.1 to write it in terms of the Fourier transform of u. We obtain that it is bounded by

$$\int_{\mathbb{R}} |\widehat{u}(\xi)|^2 e^{4\pi\frac{(2^{1-2/q}a)^q}{q}|\xi|^q} d\xi .$$

The fact that there is a non zero function u for which both integrals are finite is an easy consequence of Theorem 1.4 and Schwarz inequality, since $\left(2^{1-\frac{2}{p}}a\right) \times \left(2^{1-\frac{2}{q}}b\right) < \left|\cos\left(\frac{p\pi}{2}\right)\right|^{\frac{1}{p}}$. $\qquad\square$

Remark : At this stage, we would like to point out that we have not been able to weaken the condition $\iint \frac{|A(u,v)(x,y)|^2}{(1+\|x\|+\|y\|)^N} e^{\pi(\|x\|^2+\|y\|^2)} dxdy < +\infty$ in Theorem 6.1 into the condition

$$\iint \frac{|A(u,v)(x,y)|^2}{(1+\|x\|+\|y\|)^N} e^{2\pi|\langle x,y\rangle|} dxdy < +\infty \text{ or at least } \iint \frac{|A(u,v)(x,y)|^2}{(1+\|x\|+\|y\|)^N} e^{2\pi\|x\|\|y\|} dxdy < +\infty.$$

Also, we do not know whether weaker conditions, with $|A(u,v)(x,y)|$ in place of its square, are sufficient.

APPENDIX

We give here a simplified proof of Beurling-Hörmander's Theorem, which may be useful for elementary courses on Fourier Analysis. All ideas are contained in Section 2.

We want to prove that a function $f \in L^1(\mathbb{R})$ which satisfies the inequality

(39)
$$\iint_{\mathbb{R}\times\mathbb{R}} |f(x)||\widehat{f}(\xi)| e^{2\pi|x||\xi|} dxd\xi < +\infty$$

is identically 0. It is sufficient to show that the function $g = e^{-\pi x^2} * f$ is identically 0. Indeed, the Fourier transform of g is equal to $e^{-\pi\xi^2}\widehat{f}$. If it is 0, then f vanishes also. Now g extends to an entire function of order 2 in the complex plane. We claim that, moreover,

(40)
$$\iint_{\mathbb{R}\times\mathbb{R}} |g(x)||\widehat{g}(\xi)| e^{2\pi|x||\xi|} dxd\xi < +\infty$$

Indeed, replacing g and $\widehat{g}$ by their values in terms of f and $\widehat{f}$ and using Fubini's theorem, we are led to prove that the quantity

$$\int_{\mathbb{R}} e^{-\pi[(x-y)^2 - 2|x||\xi| + 2|y||\xi| + \xi^2]} dx$$

is bounded independently of y and ξ. Taking $x - y$ as the variable, it is sufficient to prove that

$$\int_{\mathbb{R}} e^{-\pi[x^2 - 2|x||\xi| + \xi^2]} dx$$

is bounded by 2, which follows from the fact that $x^2 - 2|x||\xi| + \xi^2$ is either $(x-\xi)^2$ or $(x+\xi)^2$.

Now, for all $z \in \mathbb{C}$, we have the elementary inequality

$$|g(z)| \le \int_{\mathbb{R}} |\widehat{g}(\xi)| e^{2\pi|z||\xi|} d\xi,$$

so that there exists some constant C such that

(41)
$$\int_{-\infty}^{+\infty} |g(x)| \sup_{|z|=|x|} |g(z)| dx \le C.$$

We claim that the holomorphic function

$$G(z) = \int_0^z g(u)g(iu)du$$

is bounded by C. Once we know this, the end of the proof is immediate: G is constant by Liouville's Theorem; so $g(u)g(iu)$ is identically 0, which implies that g is identically 0.

It is clear from (41) that G is bounded by C on the axes. Let us prove that it is bounded by C for $z = re^{i\theta}$ in the first quadrant. Assume that θ is in the interval $(0, \pi/2)$. By continuity, it is sufficient to prove that

$$G_\alpha(z) = \int_0^z g(e^{-i\alpha}u)g(iu)du$$

is bounded by C for all $\alpha \in (0,\theta)$. But the function G_α is an entire function of order 2, which is bounded by C on the y-axis and on the half-line $\rho e^{i\alpha}$. By Phagmèn–Lindelhöf principle, it is bounded by C inside the angular sector, which gives the required bound for $|G_\alpha(z)|$. A similar proof gives the same bound in the other quadrants.

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